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Random Determinants[1]

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Determinants whose elements involve random variables are discussed and expressions derived for the first and second moments. Applications are made to n-dimensional geometry, especially, to finding limiting probabilities for the event: "a given point lies above a random hyperplane", under fairly general hypotheses. The random variable $A_n h-B$ is considered, where A_n and B are certain minors of the determinantal equation of the random hyperplane, and h is a coordinate of the given point.

An asymptotic expression for $E\{A_n^2\}$ is obtained, and it is shown that $E\{B^2\}$ is of the order of $(1/n)\ E\{A_n^2\}$.

1.

In this section real random variables X_{ij} depending on two indices i and j and having the following properties are considered:

Let A_1, A_2, . . ., A_k be k random variables chosen from the X_{ij}'s. Then we make the assumptions that

(a) for any i and j, for any k, and for any choice of the A_l's, the conditional mathematical expectation

$$E(X_{ij}|A_1,\ \ldots,\ A_k)=0,$$

if X_{ij} is not one of the A_l's.

(b) For any i, j, for any k, and for any choice of the A_l's, the conditional second order moment

$$E(X_{ij}^2|A_1,\ \ldots,\ A_k)=\sigma_i^2,$$

(where σ_i is a constant that depends only on i) provided X_{ij} is not one of the A_l's.

Consider the following random determinant:

$$\Delta_n=\begin{vmatrix} X_{11} & X_{12} & \ldots & X_{1n} \\ X_{21} & X_{22} & \ldots & X_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ X_{n1} & & \ldots & X_{nn} \end{vmatrix}$$

We have the

Lemma: $E(\Delta_n)=0$ *and* $E(\Delta_n^2)=n!\prod_{i=1}^{n}\sigma_i^2$.

Proof. Let A_{ij} be the cofactor of X_{ij} in Δ_n; we can write

$$\Delta_n=\sum_j A_{1j}X_{1j}$$

$$E(\Delta_n)=\sum_j E\{A_{1j}E(X_{1j}|A_{1j})\}.$$

From (a), $E(X_{1j}|A_{1j})=0$; hence

$$E(\Delta_n)=0.$$

Also

$$\Delta_n^2=\sum_j A_{1j}^2X_{1j}^2+\sum_{j\neq k}A_{1j}A_{1k}X_{1j}X_{1k}.$$

Now

$$E\{\sum_{j\neq k}A_{1j}A_{1k}X_{1j}X_{1k}\}=$$
$$\sum_{j\neq k}E\{A_{1j}A_{1k}X_{1j}E(X_{1k}|A_{1j},A_{1k},X_{1j})\}$$

From (a) $E(X_{1k}|A_{1j},A_{1k},X_{1j})=0$; hence

$$E\{\Delta_n^2\}=\sum_j E\{A_{1j}^2X_{1j}^2\}=\sum_j E\{A_{1j}^2E(X_{1j}^2|A_{1j})\},$$

and from (b) this reduces to

$$E\{\Delta_n^2\}=\sigma_1^2\sum_j E(A_{1j}^2). \tag{1}$$

Now A_{1j} is a Δ_{n-1}, which depends only on those X_{ij} with $i\neq 1$. If we assume that the formula

$$E\{\Delta_n^2\}=n!\sum_{i=1}^{n}\sigma_i^2 \tag{2}$$

is true for $n=1, 2, \ldots, r$, (1) shows that (2) is also valid for $n=r+1$; on the other hand (2) obviously holds for $n=1$. Thus (2) is established by induction, and the proof of the lemma is complete.

$$d_n=\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & \cdot & \ldots & x_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_{n1} & \cdot & \ldots & x_{nn} \end{vmatrix}.$$

Nonrandom real numbers x_{1j} and the determinant are now considered.

Suppressing the k rows i_1, i_2, . . . , i_k in d_n, a rectangular matrix is obtained with $(n-k)$ rows; $T[i_1,i_2,\ \ldots\ ,i_k;d_n]$ is called the sum of the squares of all determinants of $(n-k)$th order deduced from this matrix.

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Let D_n be the following random determinant:

$$D_n = \begin{vmatrix} x_{11}+X_{11} & x_{12}+X_{12} & \ldots & x_{1n}+X_{1n} \\ x_{21}+X_{21} & \cdot & & \ldots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_{n1}+X_{n1} & x_{n2}+X_{n2} & \ldots & x_{nn}+X_{nn} \end{vmatrix}$$

D_n can be written as a sum of elementary determinants, the elements of which are either x_{ij} or X_{ij} but never $x_{ij}+X_{ij}$. One of these elementary determinants is d_n; all the other elementary determinants are random determinants. Let S be one of these random determinants: S consists of k rows of X_{ij}'s ($k \geq 1$) and $(n-k)$ rows of x_{ij}'s. For instance, suppose that the first row of S consists of the X_{1j}; S is a linear homogeneous form in the X_{1j}'s, the coefficients of which are independent of the X_{1j}'s. Thus according to (a) $E(S)=0$; consequently $E(D_n)=d_n$.

Let $i_1, i_2, \ldots, i_k$ be the k rows of S consisting of some X_{ij}'s, that is, row i_1 consists of the $X_{i_1 j}$, row i_2 of the $X_{i_2 j}$, etc. Employing the Laplace development of S in terms of these k rows $i_1, i_2, \ldots, i_k$, we can write S in the form:

$$S = \Sigma^* \Delta_k(j_1,j_2, \ldots, j_k) B(j_1,j_2, \ldots, j_k),$$

where $\Delta_k(j_1,j_2, \ldots, j_k)$ is a determinant of the preceding Δ type, the elements of which are those X_{ij} with $i \in i_1, i_2, \ldots, i_k$, $j \in j_1, \ldots, j_k$; $B(j_1,j_2, \ldots, j_k)$ is the algebraic complement of $\Delta_k(j_1,j_2, \ldots, j_k)$; and the summation Σ^* is extended over all combinations $(j_1,j_2, \ldots, j_k)$ of order k taken from the n integers $1, 2, \ldots, n$. For two different combinations we have:

$$E\{\Delta_k(j_1,j_2, \ldots, j_k)\cdot\Delta_k(j_1', \ldots, j_k')\}=0, \quad (3)$$

since there is at least one j_α, j_1 say, which is not a j_β', so that the product $\Delta_k(j_1, \ldots, j_k)\cdot\Delta_k(j_1', \ldots, j_k')$ is a homogeneous linear form in $X_{i_1 j_1}, X_{i_2 j_1}, \ldots, X_{i_k j_1}$, and (3) follows from (a). Consequently

$$E(S^2) = \Sigma^* B^2(j_1, \ldots, j_k) E\{\Delta_k^2(j_1, \ldots, j_k)\},$$

and by the lemma,

$$E(S^2) = \sum{}^* (B^2(j_1, \ldots, j_k)\cdot k! \prod_{\alpha=1}^{k} \sigma^2_{i_\alpha})$$
$$= k! \prod_{\alpha=1}^{k} \sigma^2_{i_\alpha} [\sum{}^* B^2(j_1, \ldots, j_k)].$$

But obviously,

$$\Sigma^* B^2(j_1, \ldots, j_k) = T(i_1,i_2, \ldots, i_k; d_n).$$

Hence we have

$$E(S^2) = T(i_1,i_2, \ldots, i_k; d_n) k! \prod_{\alpha=1}^{k} \sigma^2_{i_\alpha}.$$

Consider the product of two different elementary determinants S_1 and S_2. There is at least one row consisting of some X_{ij} that appear in S_1 and not in S_2, or in S_2 and not in S_1. For example, suppose the first row in S_1, consisting of the X_{1j}'s, does not appear in S_2; then the product S_1S_2 is a linear homogeneous form in the X_{1j}'s, the coefficients of which are independent of the X_{1j}'s. Consequently by (a), $E(S_1S_2)=0$ and $E(D_n^2)$ reduces to the sum of the squares of all the elementary determinants. This gives us the following:

Theorem 1: Under assumptions (a) *and* (b),

$$E(D_n) = d_n \quad (4)$$

$$E(D_n^2) = \sum_{k=0}^{n} k! (\sum{}^* T(i_1,i_2, \ldots, i_k; d_n) \prod_{\alpha=1}^{k} \sigma^2_{i_\alpha}). \quad (5)$$

In (5) the summation Σ^* is extended over all combinations $(i_1, i_2, \ldots, i_k)$ of order k of the integers $1, 2, \ldots, n$; for convenience we put

$$T(1,2, \ldots, n; d_n) = 1.$$

An interesting feature of formulas (4) and (5) is that these formulas do not depend on the probability laws of the X_{ij}'s. This fact remains valid even if the σ_i's depend on j, but (5) becomes more complicated; on the other hand it is possible to compute the higher moments of D_n by using similar reasoning and suitable assumptions. However, the formulas seem quite complicated. It also appears difficult to obtain the probability law of D_n, even under such hypotheses as that the X_{ij} are normally distributed.

Of course if $\sigma_1=\sigma_2= \ldots =\sigma_n=\sigma$, where σ is a constant independent of i, (5) reduces to:

$$E(D_n^2) = \sum_{k=0}^{n} T_k(d_n) k! \sigma^{2k}, \quad (5a)$$

where

$$T_k(d_n) = \Sigma^* T(i_1, \ldots, i_k; d_n).$$

From (5) certain interesting results may be deduced, which are perhaps already known. These results are connected with the geometrical interpretation of the coefficients of (5). Let E_n be an n-dimensional Euclidean space, with orthogonal coordinates. The n numbers $x_{i1}, x_{i2}, \ldots, x_{in}$ may be regarded as the coordinates of a point P_i in E_n, and $x_{i1}+X_{i1}, x_{i2}+X_{i2}, \ldots, x_{in}+X_{in}$ can be considered as the coordinates of a random point M_n in E_n. That d_n is invariant under orthogonal transformations is well known. If $(0,P_1P_2 \ldots P_n)$ is the volume in E_n of the parallelepiped formed by the vectors $\overrightarrow{0P_1}, \overrightarrow{0P_2}, \ldots, \overrightarrow{0P_n}$ we have:

$$d_n = \pm(0, P_1P_2 \ldots P_n).$$

An analogous interpretation holds for D_n and $(0, M_1M_2 \ldots M_n)$; hence $E(D_n^2)$ is invariant under an orthogonal transformation. If the X_{ij}'s are normally distributed, this property also persists. The

σ_i remain unaltered also. We conclude that the coefficients $T(i_1, \ldots, i_k; d_n)$ are invariant.

$T(i_1, \ldots, i_k; d_n)$ depends only on $P_{i_1}, \ldots, P_{i_k}$; consider the set of points $P'_{i_1}, P'_{i_2}, \ldots, P'_{i_k}$ defined as follows:

$$\overrightarrow{0P'_{i_1}} = \overrightarrow{0P_{i_1}}$$

$$\overrightarrow{0P'_{i_2}} = \overrightarrow{0P_{i_2}} + \lambda_{21}\overrightarrow{0P_{i_1}}$$

$$\vdots$$

$$\overrightarrow{0P'_{i_k}} = \overrightarrow{0P_{i_k}} + \lambda_{k1}\overrightarrow{0P_{i_1}} + \ldots + \lambda_{k,\,k-1}\overrightarrow{0P_{i_{k-1}}},$$

where the λ_{ij} are such that the vectors $\overrightarrow{0P'_{i_1}}, \overrightarrow{0P'_{i_2}}, \ldots, \overrightarrow{0P'_{i_k}}$ are mutually orthogonal. From a classical property of determinants we can replace $P_{i_1}, \ldots, P_{i_k}$ by $P'_{i_1}, \ldots, P'_{i_k}$ without altering $T(i_1, \ldots, i_k; d_n)$. The invariance of $T(i_1, \ldots, i_k; d_n)$ under orthogonal transformations implies that the vectors $\overrightarrow{0P'_{1_\alpha}}$ ($\alpha=1, \ldots, k$) can be taken as coordinate axes. Then it becomes obvious that

$$T(i_1, \ldots, i_k; d_n) = (0, P'_{i_1} \ldots P'_{i_k})^2,$$

the volume $(0, P'_{i_1} \ldots P'_{i_k})$ being considered as a volume in a k-dimensional subspace of E_n; but it is also obvious that

$$(0, P'_{i_1} \ldots P'_{i_k})^2 = (0, P_{i_1} P_{i_2} \ldots P_{i_k})^2,$$

and we have established:

Theorem 2:

$$T(i_1, \ldots, i_k; d_n) = (0, P_{i_1} \ldots P_{i_k})^2, \tag{6}$$

and consequently,

$$T_k(d_n) = \sum{}^*(0, P_{i_1} \ldots P_{i_k})^2. \tag{7}$$

On the other hand, considering $P_1, \ldots, P_n$ as fixed points in E_n, and Ω as a moving point, $(\Omega, P_{i_1} \ldots P_{i_k})^2$ and $\Sigma^*(\Omega, P_{i_1} \ldots P_{i_k})^2$ are given by (6) or (7) under a simple change of the origin of coordinates. If one puts

$$(\Omega, P_{i_1} \ldots P_{i_k})^2 = C \tag{8}$$

and

$$\Sigma^*(\Omega, P_{i_1} \ldots P_{i_k})^2 = C^*, \tag{9}$$

where C and C^* are any positive constants, then eq (8) and (9) define quadrics $Q(i_1, \ldots, i_k; C)$ and $Q_k(C^*)$, respectively, of ellipsoidal type. Thus it follows from (5) that the relation

$$E\{(\Omega, M_1 \ldots M_n)^2\} = e, \tag{10}$$

where e is some positive constant, also defines a quadric $Q(e)$ (which is in general an ellipsoid).

2.

In Euclidean n-space with the coordinates defined as above consider n fixed points $P_1, P_2, \ldots, P_n$, and a fixed point S on the axis $0x_n$ with coordinates $(0, 0, \ldots, 0, h)$. Let $M_1, M_2, \ldots, M_n$ be n random points, and consider the random plane Π, (i. e., an $(n-1)$-dimensional linear variety) defined by $(M_1, M_2, \ldots, M_n)$. The equation of Π has the following determinantal form

$$U = \begin{vmatrix} x_1 & x_2 & \ldots & x_{n-1} & x_n & 1 \\ x_{11}+X_{11} & x_{12}+X_{12} & \ldots & x_{1,\,n-1}+X_{1,\,n-1} & X_{1n} & 1 \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ x_{n1}+X_{n1} & \ldots & & x_{n,\,n-1}+X_{n,\,n-1} & X_{nn} & 1 \end{vmatrix} = 0$$

Developing this determinant in terms of the elements of the first row and their cofactors, one obtains

$$U = \sum_{j=1}^{n} A_j x_j - B = 0. \tag{11}$$

For convenience we set

$$Y_{ij} = x_{ij} + X_{ij} \qquad (i=1, 2, \ldots, n;\ j=1, 2, \ldots, n-1).$$

Let M'_i be the point $(Y_{i2}, \ldots Y_{i,n-1}, 0)$ and V the linear variety of $n-2$ dimensions defined by

$$x_n = 0, \qquad U = 0, \tag{12}$$

and let H_i be the orthogonal projection of M_i or M'_i on V. An elementary computation shows that, in absolute value,

$$\frac{\overline{M'_i M_i}}{\overline{H_i M'_i}} = \frac{|X_{in}|}{\overline{H_i M'_i}} = \frac{\sqrt{\sum_{j=1}^{n-1} A_j^2}}{|A_n|} \tag{13}$$

We now suppose that:

(A) the distance between P_i and P_j is equal to 1, for any i and j. In this case the n points P_i form a regular polyhedron in the variety $x_n=0$; this polyhedron admits $\binom{n}{k}$ linear varieties V_{ij} as varieties of symmetry, V_{ij} being defined as that $(n-2)$-dimensional variety formed by the points in $x_n=0$ which are equidistant from P_i and P_j $(i \neq j)$. Now suppose that V is a moving $(n-2)$-dimensional variety, and consider the quantity

$$s = \max_i \text{ (distance from } V \text{ to } P_i).$$

s attains its minimum value when V is a V_{ij}, in which case $s=\frac{1}{2}$. Consequently in all cases there is at least one point P_i such that the distance from V to P_i is at least $\frac{1}{2}$.

Now suppose that n tends to $+\infty$ with condition (A) always holding; we make the following assumptions:

(B) $\lim_{n\to\infty} Pr\left\{\operatorname{Max} |X_{in}|<\epsilon\right\}=1$ for any $\epsilon>0$;

(C) there exists a positive number $a<\frac{1}{2}$ and independent of n such that:

$$\lim_{n\to\infty} Pr\left\{\operatorname*{Max}_i \sum_{j=1}^{n} X_{ij}^2<a^2\right\}=1.$$

If for instance the distance from P_i to V is a maximum when $i=1$, then except in cases of small probability:

$$\overline{H_1M_1'}>\frac{1}{2}-a,$$

and

$$|X_{1n}|<\epsilon.$$

Thus we have:

Theorem 3: Under hypotheses (A), (B), *and* (C),

$$\lim_{n\to\infty} Pr\left\{\frac{\sqrt{\sum_1^{n-1} A_j^2}}{|A_n|}>0\right\}=0.$$

Remark: Assumptions (B) and (C) are not particularly restrictive. They are satisfied, for instance if the X_{ij}'s are independent and normally distributed with

$$E(X_{ij}^2)=\frac{\rho^2}{n}, \tag{14}$$

ρ being $<\frac{1}{2}$ and independent of i, j, n; also when the M_i are independently distributed with uniform probability density over the interior of the sphere of center P_i and radius $\rho<\frac{1}{2}$.

Denote the center of gravity of $P_1, \ldots, P_n$ by G, with coordinates $(\xi_1, \ldots, \xi_n)$ given by

$$\xi_j=\frac{1}{n}\sum_{i=1}^{n} x_{ij}, \qquad (j=1, 2, \ldots, n), \tag{15}$$

$\xi_n=0$. Let K be the center of gravity of $M_1, \ldots, M_n$ with coordinates ξ_j+Z_j $(j=1,2, \ldots, n)$. Put

$$Z_j=\frac{1}{n}\sum_{i=1}^{n} X_{ij}. \tag{16}$$

We make the following assumptions:

(D) the distance $\overline{OG}$ remains bounded as $n\to+\infty$;

(E) the X_{ij} satisfy assumptions (a) and (b) of section 1, with

$$\sigma_i\leq\frac{\rho}{\sqrt{n}},$$

where ρ^2 is a constant independent of i and n. The algebraic distance L from G to Π is equal to:

$$L=\frac{\sum_{j=1}^{n-1} A_j\xi_j-B}{\sqrt{\sum_{j=1}^{n} A_j^2}}. \tag{17}$$

K belongs to Π, so that

$$|L|\leq\overline{GK}.$$

But the coordinates of the vector $\overrightarrow{GK}$ are the Z_j's defined by (16). From (E) it then follows that

$$E(\overline{GK}^2)\leq\frac{\rho^2}{n}.$$

Hence $\overline{GK}$ and *a fortiori* L tend toward zero in probability. Write (17) as

$$\frac{\sum_{j=1}^{n-1} A_j\,\xi_j-B}{\sqrt{\sum_{j=1}^{n} A_j^2}}=-\frac{B}{A_n}\cdot\frac{A_n}{|A_n|\sqrt{1+\frac{\sum_1^{n-1} A_j^2}{|A_n|^2}}}+\frac{\sum_{j=1}^{n-1} A_j\,\xi_j}{|A_n|\sqrt{1+\frac{\sum_1^{n-1} A_j^2}{|A_n|^2}}}.$$

We have by the Schwartz inequality

$$\left|\sum_{j=1}^{n-1} A_j\xi_j\right|\leq\sqrt{\sum_{j=1}^{n-1} A_j^2}\cdot\sqrt{\sum_{j=1}^{n-1} \xi_j^2},$$

so that

$$\frac{\left|\sum_{j=1}^{n-1} A_j\xi_j\right|}{|A_n|\sqrt{1+\frac{\sum_1^{n-1} A_j^2}{|A_n|^2}}}\leq\sqrt{\sum_{j=1}^{n-1} \xi_j^2}\,\frac{\frac{\sqrt{\sum_1^{n-1} A_j^2}}{|A_n|}}{\sqrt{1+\frac{\sum_1^{n-1} A_j^2}{|A_n|^2}}}.$$

Now $\sqrt{\sum_{j=1}^{n-1} \xi_j^2}$ is bounded and $\sqrt{\sum_1^{n-1} A_j^2}\Big/|A_n|$ tends to zero in probability. Therefore under the assumptions (A)—(E) we have:

Lemma: As $n\to\infty$, $|B/A_n|$ *tends to zero in probability.*

The algebraic distance from S to Π is

$$\Delta=\frac{h A_n-B}{\sqrt{\sum_{j=1}^{n} A_j^2}}=\frac{h A_n-B}{|A_n|\sqrt{1+\frac{\sum_1^{n-1} A_j^2}{|A_n|^2}}}$$

$$=\left(h\frac{A_n}{|A_n|}-\frac{B}{|A_n|}\right)\frac{1}{\sqrt{1+\frac{\sum_1^{n-1} A_j^2}{|A_n|^2}}}. \tag{18}$$

Suppose that $h>0$; then S is above Π if $\Delta A_n>0$, or what is the same thing, $(hA_n^2-BA_n)>0$. We may write this last expression as

$$h-\frac{B}{A_n}>0.$$

Thus according to the above lemma (and under the same hypotheses, viz., assumption (A)-(E)):

Theorem 4. If $h>0$, the probability that S is above Π *tends to 1 when $n\to+\infty$; if $h<0$, this probability tends to zero.*

The case $h=0$ is not clear-cut. But if, for instance, the M_i's are independent with the same continuous probability law which is symmetric about the plane $x_n=0$, then clearly $Pr\{S>\pi\}=\frac{1}{2}$.

3.

Keeping the notations of section 2, we assume that $\overline{P_i P_j}=1$ for any i, j and that the X_{ij}'s satisfy conditions (a) and (b) of section 1 with $\sigma_1=\ldots=\sigma_n=\rho/\sqrt{n}$ where ρ is a constant independent of i and n. The other assumptions, (B), (C), and (D) of section 2, will not be required. We wish to obtain some information about the random variable:

$$H=A_n h-B. \tag{19}$$

Theorem 1 implies immediately that

$$E(H)=V_n h, \tag{20}$$

where V_n is the determinant

$$V_n=\begin{vmatrix} x_{11} & x_{12}\ldots x_{1,n-1} & 1 \\ x_{21} & x_{22}\ldots x_{2,n-1} & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_{n1} & x_{n2}\ldots x_{n,n-1} & 1 \end{vmatrix}.$$

V_n, as we have seen, is given by

$$V_n=\pm(P_1, P_2P_3\ldots P_n).$$

That is to say (cf. formula (33) of the Appendix)

$$V_n=\pm\sqrt{\frac{n}{2^{n-1}}}. \tag{21}$$

Let w_n be the determinant

$$\begin{vmatrix} x_{11} & x_{12}\ldots x_{1,n-1} & 0 \\ x_{21} & \ldots x_{2,n-1} & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_{n1} & \ldots x_{n,n-1} & 0 \end{vmatrix}.$$

For $E(H^2)$ we have at once:

$$E(H^2)=h^2E(A_n^2)+E(B^2). \tag{22}$$

Since B is a determinant of the form D_n (section 1) formula (5*a*) gives the result that

$$E(B^2)=\sum_{k=1}^{n} T_k(w_n)\,k!\frac{\rho^{2k}}{n^k}. \tag{23}$$

A_n is not exactly of type D_n, but in entirely similar fashion we find

$$E(A_n^2)=\sum_{k=0}^{n-1} T_k'(V_n)\,k!\frac{\rho^{2k}}{n^k}. \tag{24}$$

$T_k'(V_n)$ is the sum of the minors of order $(n-k)$ of V_n obtained without suppressing the last column of V_n. For instance

$$T_k'(i_1,\ldots,i_k;V_n)=(P_{j_1}, P_{j_2}P_{j_3}\ldots P_{j_{n-k}})^2,$$

where $j_1, j_2,\ldots,j_{n-k}$ run through all the numbers $1, 2,\ldots,n$ except $i_1, i_2,\ldots,i_{n-k}$. Hence

$$T_k'(i_1,\ldots,i_k;\,V_n)=V_{n-k}^2=\frac{n-k}{2^{n-k-2}} \tag{25}$$

and

$$T_k'(V_n)=\binom{n}{k}(n-k)\,2^{-n+k+1}. \tag{26}$$

Substituting this expression into (24) one finds

$$E(A_n^2)=\sum_{k=0}^{n-1}\binom{n}{k}(n-k)2^{-n+k+1}\cdot k!\frac{\rho^{2k}}{n^k}$$

$$=\frac{n}{2^{n-1}}\sum_{k=0}^{n-1}\frac{(n-1)\ldots(n-k)}{n^k}(2\rho^2)^k. \tag{27}$$

If we suppose that $2\rho^2<1$, (27) yields immediately:

$$E(A_n^2)\sim\frac{1}{1-2\rho^2}\cdot\frac{n}{2^{n-1}}, \tag{27a}$$

as $n\to\infty$.

B is of the form D_n; we know that $Q_k(C^*)$ is a quadric, and it must have as planes of symmetry the $\binom{n}{2}$ planes of symmetry V_{ij} of the polyhedron

$(P_1, P_2, \ldots, P_n)$. Consequently the center of $Q_k(C^*)$ is G, and $Q_k(C^*)$ is a sphere since $\binom{n}{2} > n$, and we can write:

$$T_k(w_n)=\lambda_k(n)\overline{OG}^2+T_k^*(w_n), \tag{28}$$

where $T_k^*(w_n)$ is the value of $T_k(w_n)$ when $\overline{OG}=0$. On the other hand

$$T_k(i_1, \ldots, i_k; w_n)=(O, P_{j_1}P_{j_2} \ldots P_{j_{n-k}}),$$

where $j_1, j_2, \ldots, j_{n-k}$ are all the numbers $1, 2, \ldots, n$ except $i_1, i_2, \ldots, i_k$. If $O=G$, we find (cf. appendix, formula (36)):

$$T^*(i_1, \ldots, i_j; w_n)=\frac{k}{n2^{n-k}}$$

and

$$T_k^*(w_n)=\binom{n}{k}\frac{k}{n2^{n-k}}.$$

Hence

$$\sum_{k=1}^{n} T_k^*(w_n)k!\frac{\rho^{2k}}{n^k}=\frac{1}{n2^n}\sum_{k=1}^{n} k\frac{n(n-1)\ldots(n-k+1)}{n^k}(2\rho^2)^k. \tag{29}$$

If $2\rho^2<1$, (29) becomes asymptotically

$$\sum_{k=1}^{n} T_k^*(w_n)k!\frac{\rho^{2k}}{n^k}\sim\frac{1}{n2^n}\cdot\frac{1\rho^2}{(1-2\rho^2)^2}. \tag{29a}$$

We can now compute the value $T_k^{**}(w_n)$ of $T_k(w_n)$ for $P_n=O$ (cf. appendix, formula (38)). Writing

$$T_k^{**}(w_n)=\lambda_k(n)\overline{GP}_n^2+T_k^*(w_n),$$

we have an equation which determines $\lambda_k(n)$; we find

$$\lambda_k(n)=\frac{n(n-2)(n-3)\ldots(n-k)}{(k-1)!2^{n-k-1}}$$

and

$$\sum_{k=1}^{n}\lambda_k(n)k!\frac{\rho^{2k}}{n^k}=\frac{1}{2^{n-1}}\sum_{k=1}^{n}\frac{n(n-2)(n-2)\ldots(n-k)}{n^k}k(2\rho^2)^k. \tag{30}$$

Asymptotically this expression becomes, for $2\rho^2<1$,

$$\sum_{k=1}^{n}\lambda_k(n)k!\frac{\rho^{2k}}{n^k}\sim\frac{1}{2^{n-1}}\frac{2\rho^2}{(1-2\rho^2)^2}. \tag{30a}$$

We verify that $E(B^2)$ is of the order of $(1/n)E(A_n^2)$, which agrees with the lemma of section 2. In the particular case when $G=O$, $E(B^2)$ is of the order of $(1/n^2)\,E(A_n^2)$.

Appendix

Given n points $P_1, \ldots, P_n$ in an $(n-1)$-dimensional Euclidean space E_{n-1} such that $\overline{P_iP_j}=1$ for any i and j $(i\neq j)$ (that is, these points are the vertices of a regular polyhedron $\boldsymbol{P}_n$) we wish to find an expression for the volume $(1/n!)\,|V_n|$ of $\boldsymbol{P}_n$ in E_{n-1}. Let G_i be the center of gravity of the points $P_1, P_2, \ldots, P_i$; h_n the distance between G_{n-1} and P_n; and d_n the distance between G_n and any P_i. We have

$$|V_n|=h_n|V_{n-1}|$$

$$(n-1)\ \overrightarrow{G_nG_{n-1}}+\overrightarrow{G_nP_n}=0. \tag{31}$$

G_{n-1}, G_n, P_n are on the same "straight line" (one-dimensional linear variety); and $P_1G_{n-1}P_n$ is a right triangle; hence:

$$1=\overline{P_1P}_n^2=\overline{P_1G}_{n-1}^2+\overline{G_{n-1}P}_n^2.$$

It follows that

$$d_n=\frac{n-1}{n}h_n$$

$$h_n^2=1-\left(\frac{n-2}{n-1}\right)^2h_{n-1}^2. \tag{32}$$

Clearly h_3 is known; the recurrence equation (32) can be solved for h_n:

$$h_n=\frac{1}{\sqrt{2}}\sqrt{\frac{n}{n-1}}.$$

It follows by recurrence from (31), $|V_3|$ being equal to $\sqrt{3}/2$. that:

$$|V_n|=\sqrt{\frac{n}{2^{n-1}}}, \tag{33}$$

On the other hand,

$$d_n=\frac{1}{\sqrt{2}}\sqrt{\frac{n-1}{n}}, \tag{34}$$

and

$$\overline{G_nG}_{n-1}^2=\frac{1}{2(n-1)n}=\frac{1}{2}\left(\frac{n}{n-1}-\frac{1}{n}\right). \tag{35}$$

We now compute the volume $(G_n, P_1P_2 \ldots P_{n-k})$. If L_{nk} is the orthogonal projection of G_n on the linear variety $(P_1P_2 \ldots P_{n-k})$, we have:

$$(G_n, P_1P_2 \ldots P_{n-k})^2=\overline{G_nL}_{nk}^2|V_{n-k}|^2.$$

But $L_{nk}=G_{n-k}$, so that

$$\overline{G_nL}_{nk}^2=\overline{G_nG}_{n-1}^2+\overline{G_{n-1}G}_{n-2}^2+\ \ldots\ +\overline{G_{n-k+1}G}_{n-k}^2.$$

As in (35),

$$\overline{G_nL}_{nk}^2=\frac{1}{2}\left(\frac{1}{n-k}-\frac{1}{n}\right)=\frac{k}{2n(n-k)},$$

and so

$$(G_n, P_1 \ldots P_{n-k})^2=\frac{k}{n2^{n-k}}. \tag{36}$$

We can also compute $(P_n, P_1P_2 \ldots P_{n-k})$:

$$(P_n, P_1P_2 \ldots P_{n-k})^2=\overline{P_nL}_{nk}^2|V_{n-k}|^2$$

$$\overline{P_nL}_{nk}^2=\overline{P_nG}_{n-1}^2+\overline{G_{n-1}G}_{n-2}^2+\ \ldots\ +\overline{G_{n-k+1}G}_{n-k}^2$$

$$=\frac{1}{2}\left(1+\frac{1}{n-k}\right).$$

Therefore

$$(P_n, P_1 \ldots P_{n-k})^2=\frac{(n-k+1)}{2^{n-k}}. \tag{37}$$

From (37) it follows that

$$T_k^{**}(w_n)=\binom{n-1}{k-1}\frac{n-k+1}{2^{n-k}}, \tag{38}$$

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